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Lesson Plan

1. Puzzle: Discuss the puzzle from last week. Offer a hint if necessary.
2. Linear Algebra:
   (a) Linear Algebra Review: Review some basic linear algebra concepts through talking and looking at some textbook problems. Try some proofs.
   (b) More Linear Algebra: Try introducing more advanced linear algebra.
3. Probabilty Theory: Introduce student to probabilty theory.
4. Homework: Assign homework, as well as a new puzzle.

Last Week's Puzzle

Single File Hat Executioners Puzzle. Hint:

Linear Algebra

Review of Basic Concepts

The first half of linear algebra is concerned with solving the matrix equation $Ax = b$. Here are some terms that should be covered during the course of studying this problem.

topic	**notes and problems**
vector spaces and subspaces	some possible subspaces in R^2
linear transformations, kernel, image	pf. that NS and CS are subspaces
basis and dimension	show that something is a basis for R^3
full rank, 1-1, onto, skinny and fat	real life = rectangular matrices
linear independence and span	how to tell if vectors are linearly independent
determinant	mention some properties of determinants
inner product and orthogonal projection	definition and intuition in R^3, NS of "inner product onto xy-plane"
orthogonal, orthonormal	how to check if some vectors are orthogonal
Gram-Schmidt	discuss the process $w_k = v_k - (v_k \cdot u_1)u_1 - \ldots - (v_k \cdot u_{k-1})u_{k-1}$
similar matrices	change of basis intuition powers of a similar transformation (induction)
eigenvalues and eigenvectors, char. poly.	e.values of similar matrices are same $\det(A) = \prod_{i=1}^{n} \lambda_i$

Key Theorem of Linear Algebra

Let $T : V \to V$ be a linear transformation of finite dimensional vector space V. Then, amazingly, all of the following are all equivalent:

- T is invertible.
- $\det(T) \neq 0$, where the determinant is defined by a choice of basis on V.
- $\ker(T) = 0$.
- If $b \in V$, then there exists a unique vector $v \in V$ such that $T(v) = b$.
- For any basis $v_1, \ldots, v_n$ of V, the image vectors $T(v_1), \ldots, T(v_n)$ are linearly independent.
- For any basis $v_1, \ldots, v_n$ of V, if S denotes the transpose linear transformation of T, then the image vectors $S(v_1), \ldots, S(v_n)$ are linearly independent.
- The transpose of T is invertible.
- All of the eigenvalues of T are nonzero.

Similarly, given an explicit matrix A corresponding to the linear transformation T after a selection of basis for V, all of the following are equivalent:

- A is invertible
- $\det(A) \neq 0$
- $\ker(A) = 0$
- $Ax = b$ can be uniquely solved for any b
- The columns of A are linearly independent
- The rows of A are linearly independent
- The transpose of A is invertible
- All the eigenvalues of A are nonzero

Possibly New Material

SPECTRAL THEOREM

- A symmetric matrix A has real eigenvalues.
- Its eigenvectors can be chosen to be orthonormal.

Thus $A = Q\Lambda Q^T$, where the orthonormal eigenvectors compose the columns of Q, and the eigenvalues are listed in the diagonal matrix Λ.

Proofs:

Assume we have a Hermitian matrix $A = A^H$. Then, each of the following statements can be proven based on the information given from their own definitions, with a few tricks:

1. $x^H Ax$ is real.
2. Every eigenvalue of a Hermitian matrix is real.
3. The eigenvectors of a Hermitian matrix are orthogonal to each other if the eigenvalues are distinct.
4. Note that if S is a matrix whose columns consist of eigenvectors, we have $S^{-1}AS = \Lambda$.
5. Lastly, for a real symmetric matrix $A = A^T$, the S in the identity above can be chosen to be an orthogonal matrix Q. (The eigenvectors are orthogonal, can be normalized, and finally are real since $(A - \lambda I)x = 0$.) Thus $A = Q\Lambda Q^T$. This is the **spectral theorem**.

We may revisit this topic if we study Fourier series later.

Probability Theory

1. Some basic ideas: counting, coin flipping, expectation, variance. Example problems we could solve. Random walks, probability zero vs. impossibility.
2. Definitions of sample space, random variable, probability measure and expectation.

Homework

1. (AIME 2001) P lies on 8y = 15x, Q lies on 10y = 3x and the midpoint of PQ is (8,6). Find the distance PQ.
2. For each of the following statements, either show that it is true, or give a counterexample.
   (a) If AB is full rank, then A and B are both full rank.
   (b) If A and B are both full rank, then AB is full rank.
   (c) If A and B have zero nullspace, then so does AB.
   (d) If A and B are onto, then so is AB.

   Hints:
3. Prove that the only possible subspaces in $\mathbf{R}^2$ are the zero vector, lines, and all of $\mathbf{R}^2$.
4. Consider the following random process: each day, Willywutang asks one random girl on a date, and is accepted with probabilty p, but is rejected with probability $1 - p$. What is the expected number of rejections that he will have to endure before being accepted by someone?
5. Puzzle: You are locked in an empty room, and are holding a transparent glass of water. The glass is a right cylinder, and it looks like it is about half full, but you are not sure if it is *exactly* half full. How can you accurately determine whether the glass is half full, more than half full, or less than half full? No rulers, writing utensils, or other equipment is available.