PURDUE UNIVERSITY MATH DEPARTMENT PROBLEM OF THE WEEK SPRING 2010, PROBLEM 8

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Problem Let A be any $n \times n$ matrix in which every entry is either $+1$ or -1. Prove that the determinant of A is divisible by 2^{n-1}.

Solution Add the first row of A to every other row of A, to create a new matrix B. Then $\det(A) = \det(B)$, and the latter $n-1$ rows of B have entries in $\{2, 0, -2\}$. Factoring 2 out of each of these $n-1$ rows, it follows that $\det(B) = 2^{n-1}\det(C)$, where the entries of C belong to $\{1, 0, -1\}$. The Leibniz formula for the determinant then verifies that $\det(C)$ is an integer. Hence, $\det(A)$ is divisible by 2^{n-1}. □

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