PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
FALL 2011, PROBLEM 1

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Problem Show that $x^{400} + x^{380} + \ldots + x^{20} + 1$ is divisible by $x^{20} + x^{19} + \ldots + x + 1$.

Solution By the geometric series equation,

$$\frac{(x^{20})^{21} - 1}{x^{20} - 1} = x^{400} + x^{380} + \ldots + x^{20} + 1$$

and

$$\frac{x^{21} - 1}{x - 1} = x^{20} + x^{19} + \ldots + x + 1.$$

Consequently, we aim to show that $\frac{x^{21}-1}{x-1}$ divides into $\frac{(x^{20})^{21}-1}{x^{20}-1}$, or equivalently, that $f(x) := (x^{20} - 1)(x^{21} - 1)$ divides into $g(x) := (x^{20\cdot 21} - 1)(x - 1)$. This holds if every root of $f(x)$ can also be found in $g(x)$ with at least the same multiplicity.

The roots of $f(x)$ consist of the 20th and 21st roots of unity, and are clearly also roots of $g(x)$, since the roots of $g(x)$ are the 420th roots of unity and $420 = 20 \cdot 21$. Because 20 and 21 do not share any common factors except for 1, it follows that all roots of $f(x)$ have multiplicity 1, except for the root $x = 1$ which has multiplicity 2. This multiplicity is met by $g(x)$, whose root of $x = 1$ also has multiplicity 2. □

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