PURDUE UNIVERSITY MATH DEPARTMENT PROBLEM OF THE WEEK FALL 2011, PROBLEM 2

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Problem Show that $\sin x \geq x - x^2/\pi$ if $0 \leq x \leq \pi$.

Solution Since both $\sin x$ and $x - x^2/\pi$ are symmetric about $x = \pi/2$, we need only verify the inequality for x in the interval $I := [0, \pi/2]$. Using integrals, the proposed inequality can be rewritten as

$$\int_0^x (\cos u) du \geq \int_0^x \left(1 - \frac{2u}{\pi}\right) du, \qquad x \in I.$$

It suffices to show that one integrand dominates the other over the integration region; that is:

$$\cos u \geq 1 - \frac{2u}{\pi}, \qquad u \in I$$

Let $f(u) = \cos u - 1 + \frac{2u}{\pi}$; then we aim to show that $f(u) \geq 0$ for $u \in \mathcal{I}$. Note that f is concave over I since $f''(u) = -\cos u$. Consequently, the minima of f must lie on the boundaries of I. By direct evaluation, $f(0) = f(\pi/2) = 0$.

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