PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
FALL 2011, PROBLEM 4

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Problem Show that if

$$u(x) = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \dots,$$
$$v(x) = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \frac{x^{10}}{10!} + \dots,$$
$$w(x) = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \dots,$$

then $u^3 + v^3 + w^3 - 3uvw = 1$.

Solution Observe that $u' = w, v' = u, w' = v$. Therefore,

$$\begin{aligned} \frac{d}{dx}\left(u^3 + v^3 + w^3 - 3uvw\right) &= 3(u^2u' + v^2v' + w^2w' - uvw' - uv'w - u'vw) \\ &= 3(u^2w + v^2u + w^2v - uvv - uuw - wvw) \\ &= 3(u^2w + v^2u + w^2v - uv^2 - u^2w - w^2v) \\ &= 0. \end{aligned}$$

Therefore the polynomial $f(x) := (u^3 + v^3 + w^3 - 3uvw)(x)$ is equal to a constant. Direct evaluation yields $f(0) = 1$, and thus $f \equiv 1$. □

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