PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
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Problem A coast artillery gun can fire at any angle of elevation between 0° and 90° in a fixed vertical plane. If muzzle velocity is constant ($= v_0$), determine the set H of points in the plane (and above the horizontal) which can be hit. (Neglect air resistance.)

Solution Let $\theta \in [0, \pi/2]$ denote the angle of elevation, g denote gravitational acceleration ($\sim 9.8\ m/s^2$), and t denote time. The kinematics equations are

$$x(t) = x = (v\cos\theta)t$$
$$y(t) = y = -\frac{1}{2}gt^2 + (v\sin\theta)t$$

Plugging $t = \frac{x}{v\cos\theta}$ from the first equation into the second equation yields

$$y = -\frac{1}{2}g\left(\frac{x}{v\cos\theta}\right)^2 + (v\sin\theta)\left(\frac{x}{v\cos\theta}\right) = -\frac{1}{2}\frac{gx^2}{v^2\cos^2\theta} + x\tan\theta$$

Thus, a point (x, y) can be hit if there exists $\theta \in [0, \pi/2]$ such that the equation above is satisfied. We can find more precise conditions under which such a θ will occur by solving for θ. Using the trigonometric identity $\sec^2\theta = 1 + \tan^2\theta$,

$$\begin{aligned} y &= -\frac{1}{2}\frac{gx^2}{v^2}(1+\tan^2\theta) + x\tan\theta \\ &= \left(-\frac{1}{2}\frac{gx^2}{v^2}\right)\tan^2\theta + x\tan\theta - \frac{1}{2}\frac{gx^2}{v^2} \\ 0 &= \left(\frac{1}{2}\frac{gx^2}{v^2}\right)\tan^2\theta - x\tan\theta + \left(\frac{1}{2}\frac{gx^2}{v^2} + y\right) \\ \tan\theta &= \frac{x \pm \sqrt{x^2 - \frac{2gx^2}{v^2}\left(\frac{gx^2}{2v^2} + y\right)}}{gx^2/v^2} = \frac{v^2 \pm \sqrt{v^4 - g^2x^2 - 2gyv^2}}{gx} \end{aligned}$$

Restricting ourselves to angles of elevation in $[0, \pi]$, note that $\theta \in [0, \pi/2]$ if and only if $\tan\theta \geq 0$. Since $x, y, g > 0$, and $v^4 \geq v^4 - g^2x^2 - 2gyv^2$, it is not possible for $\tan\theta$ to be negative. We only need to ensure that the discriminant is not negative, such that $\tan\theta$ is real. The necessary and sufficient condition for (x, y) in the first quadrant to be hittable is

$$v^4 - g^2x^2 - 2gyv^2$$

which can be rewritten as

$$\frac{v^4 - gx^2}{2gv^2} \geq y.$$

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