PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
FALL 2011, PROBLEM 7

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Problem For every integer $n \geq 2$, prove that

$$\sum_{k=1}^{n}(-1)^k k \binom{n}{k} = 0$$

where $\binom{n}{k}$ is the usual binomial coefficient.

Solution By the binomial theorem,

$$(1+x)^n = \sum_{k=1}^{n} \binom{n}{k} x^k.$$

Substituting $x \mapsto -x$,

$$(1-x)^n = \sum_{k=1}^{n} \binom{n}{k} (-1)^k x^k.$$

Taking the derivative of both sides with respect to x,

$$n(1-x)^{n-1}(-1) = \sum_{k=1}^{n} \binom{n}{k} (-1)^k k x^{k-1}.$$

Substituting $x = 1$,

$$0 = \sum_{k=1}^{n} \binom{n}{k} (-1)^k k.$$

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