PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
SPRING 2011, PROBLEM 1

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Problem Let $x_1, x_2, ..., x_{2011}$ be real numbers. For which real value(s) of c is

$$|x_1 - c| + |x_2 - c| + \ldots + |x_{2011} - c|$$

a minimum?

Solution We will show that choosing the median of numbers x_1 through x_{2011} yields the minimum.

Firstly, note that the function $f_i(c) := |x_i - c|$ is convex in c, because the absolute value function is convex. Since the objective function $f(c) := \sum_{i=1}^{2011} f_i(c)$ is a nonnegative weighted sum of convex functions, it follows that f is also convex in c. Thus, any local minimum that we find for f is also a global minimum.

We now argue that x_{1006} is a local minimum. Without loss of generality, suppose $x_1 \leq x_2 \leq \ldots \leq x_{2011}$. Set c equal to $x_{1006} + \delta$, where δ is a deviation such that that $x_{1005} \leq x_{1006} + \delta \leq x_{1007}$. Then

$$\begin{aligned}
f(x_{1006} + \delta) &= \sum_{i=1}^{2011} |x_i - (x_{1006} + \delta)| \\
&= \sum_{i=1}^{1005} ((x_{1006} + \delta) - x_i) \quad + \quad |(x_{1006} + \delta) - x_{1006}| \quad + \quad \sum_{i=1007}^{2011} (x_i - (x_{1006} + \delta)) \\
&= 1005 \cdot (x_{1006} + \delta) - \left(\sum_{i=1}^{1005} x_i \right) + |\delta| + \left(\sum_{i=1007}^{2011} x_i \right) - 1005 \cdot (x_{1006} + \delta) \\
&= \left(\sum_{i=1007}^{2011} x_i - \sum_{i=1}^{1005} x_i \right) + |\delta|.
\end{aligned}$$

Selecting $\delta = 0$ minimizes this expression. Thus $c = x_{1006}$ is a local minimum of $f(c)$ for $c \in [x_{1005}, x_{1007}]$. By convexity, it is also the global minimum. □

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Date: January 23, 2011.