# PURDUE UNIVERSITY MATH DEPARTMENT
# PROBLEM OF THE WEEK
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**Problem** Show that the boundary value problem

$$y'' + p(x)y' - \lambda^2 y = 0, \tag{1}$$

$y(a) = y(b) = 0$, $a \neq b$, $p(x)$ an arbitrary continuous function on $[a, b]$, does not have a nontrivial solution for any real $\lambda$.

**Solution** Since $p(x)$ is continuous, $y(x)$ should also be continuous.[1] Therefore, $y(x)$ achieves its maximum over the compact set $[a, b]$ at $x = c_{\max}$, and its minimum over $[a, b]$ at $x = c_{\min}$. Suppose that $y$ is non-trivial, such $y(c_{\max})$ and $y(c_{\min})$ are not both equal to zero. Then at least one of the two following counter-arguments must be applicable:

(1) If $y(c_{\max}) > 0$, then since $y'(c_{\max}) = 0$, Eq. (1) implies $y''(c_{\max}) = \lambda^2 y(c_{\max})$. However, at a local maximum we require $y''(c_{\max}) < 0$, which contradicts $\lambda^2 y(c_{\max}) > 0$.
(2) Similarly, if $y(c_{\min}) < 0$, then since $y'(c_{\min}) = 0$, Eq. (1) implies $y''(c_{\min}) = \lambda^2 y(c_{\min})$. However, at a local minimum we require $y''(c_{\min}) > 0$, which contradicts $\lambda^2 y(c_{\min}) < 0$.

□

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[1] Lacks rigorous justification.