PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
SPRING 2011, PROBLEM 12

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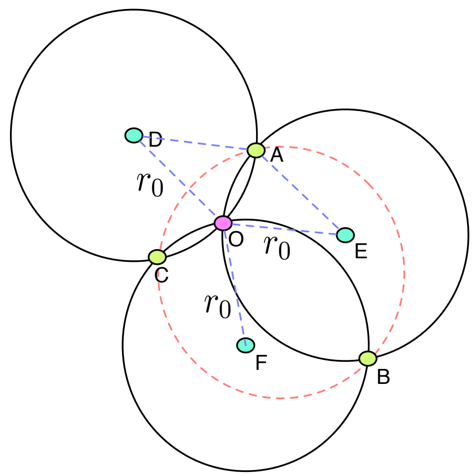
Problem Three circles in the plane, of the same radius r_0, no two of which are tangent, pass through the common point O. Show that their other points of intersection A, B, C lie on a circle of radius r_0. Hint: Use vector algebra.

Solution Consider O to be the origin in the plane. The points of intersection can be represented as vectors $\vec{A}, \vec{B}, \vec{C}$ relative to O. Let $\vec{D}, \vec{E}, \vec{F}$ denote the centroids of the circles.



Assuming that $\vec{A}$ is the intersection of the circles with centroids $\vec{D}$ and $\vec{E}$, note that the quadrilateral with corners $O, \vec{D}, \vec{E}, \vec{A}$ is a rhombus with side length r_0. Thus, by vector addition,

$$\vec{A} = \vec{D} + \vec{E}.$$

Similarly, if $\vec{B}$ is the intersection of the circles with centroids $\vec{E}$ and $\vec{F}$, and if $\vec{C}$ is the intersection of the circles with centroids $\vec{D}$ and $\vec{F}$, then

$$\vec{B} = \vec{E} + \vec{F}$$

$$\vec{C} = \vec{D} + \vec{F}.$$

Consider the circle of radius r_0 with center $\vec{D} + \vec{E} + \vec{F}$. Intersection $\vec{A}$ lies on this circle since

$$\|(\vec{D} + \vec{E} + \vec{F}) - \vec{A}\|_2 = \|(\vec{D} + \vec{E} + \vec{F}) - (\vec{D} + \vec{E})\|_2 = \|\vec{F}\|_2 = r_0.$$

Similarly, intersections $\vec{B}$ and $\vec{C}$ also lie on this circle since

$$\|(\vec{D} + \vec{E} + \vec{F}) - \vec{B}\|_2 = \|(\vec{D} + \vec{E} + \vec{F}) - (\vec{E} + \vec{F})\|_2 = \|\vec{D}\|_2 = r_0$$

$$\|(\vec{D} + \vec{E} + \vec{F}) - \vec{C}\|_2 = \|(\vec{D} + \vec{E} + \vec{F}) - (\vec{D} + \vec{F})\|_2 = \|\vec{E}\|_2 = r_0.$$

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