PURDUE UNIVERSITY MATH DEPARTMENT PROBLEM OF THE WEEK SPRING 2011, PROBLEM 13

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Problem Suppose $P(x)$ is a real polynomial of degree $k \geq 1$. Show that the power series expansion for $f(x) := e^{P(x)}$ about any point x_0, cannot have k consecutive zero coefficients.

Solution First we establish a recurrence amongst the coefficients of f. The derivative of $f(x)$ is $f'(x) = P'(x)f(x)$. Setting $u(x) = P'(x)$ and applying the general Leibniz rule (which generalizes the product rule for differentiation), the nth derivative of $f'(x)$ is given by

$$f^{(n+1)}(x) = \sum_{j=0}^{n} \binom{n}{j} u^{(j)}(x) f^{(n-j)}(x) = \sum_{j=0}^{n} \binom{n}{j} P^{(j+1)}(x) f^{(n-j)}(x).$$

Since $P(x)$ is a polynomial of degree k, $P^{(j+1)}(x) = 0$ for $j \geq k$. Therefore, for $n \geq k$,

$$f^{(n+1)}(x) = \sum_{j=0}^{k-1} \binom{n}{j} P^{(j+1)}(x) f^{(n-j)}(x).$$

So $f^{(n+1)}(x)$ is a linear combination of the k lower derivatives $f^{(n)}(x), f^{(n-1)}(x), \ldots, f^{(n-k+1)}(x)$.

We now return to the power series expansion of $f(x)$ about x_0:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

If k consecutive coefficients in this power series were zero, then k consecutive derivatives $f^{(m)}(x_0), f^{(m-1)}(x_0), \ldots, f^{(m-k+1)}(x_0)$ would be zero for some m, and by the recurrence established above, it would follow that $f^{(n)}(x_0) = 0$ for all $n \geq m - k + 1$. Then the power series for $f(x)$ about x_0 would reduce to

$$f(x) = \sum_{n=0}^{m-k} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n,$$

a polynomial of degree $m-k$. By the premise $f(x) = e^{P(x)}$, this implies that $e^{P(x)}$ is apparently a polynomial of degree $m - k$. However, this is not possible because

$$\begin{aligned} f(x) &= e^{P(x)} \\ \log f(x) &= P(x) && \text{(taking logs of both sides)} \\ \frac{f'(x)}{f(x)} &= P'(x) && \text{(differentiate both sides)} \\ f'(x) &= f(x)P'(x) \end{aligned}$$

which is a contradiction since the left-hand side is a polynomial of degree $m - k - 1$ but the right hand side is a polynomial of larger degree $\underbrace{(m-k)}_{\deg f(x)} + \underbrace{(k-1)}_{\deg P'(x)} = m - 1$. □

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