PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
SPRING 2011, PROBLEM 2

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Problem Prove that an integer whose decimal representation consists of 3^n identical digits is divisible by 3^n.

Solution Let $k \in \{1, 2, \ldots, 9\}$. Then an integer m whose decimal representation consists of 3^n identical digits can be expressed as

$$m = \sum_{i=0}^{3^n - 1} k \cdot 10^i = k \left(\frac{10^{3^n} - 1}{10 - 1} \right) = k \left(\frac{10^{3^n} - 1}{3^2} \right)$$

We will show that the parenthesized term is divisible by 3^n. This reduces to showing that $10^{3^n} - 1$ is divisible by 3^{n+2}, which we now prove by induction. For $n = 0$, $10^{3^0} - 1 = 3^{0+2}$. For the induction step, we will show that $10^{3^{n+1}} - 1$ is divisible by 3^{n+3}. Using the factorization $x^3 - 1 = (x-1)(x^2 + x + 1)$ with $x = 10^{3^n}$,

$$10^{3^{n+1}} - 1 = (10^{3^n})^3 - 1 = \underbrace{(10^{3^n} - 1)}_{\alpha_n} \underbrace{\left(\left(10^{3^n}\right)^2 + 10^{3^n} + 1 \right)}_{\beta_n}.$$

By the induction hypothesis, α_n is divisible by 3^{n+2}. And β_n is divisible by 3, since

$$\begin{aligned} \left(10^{3^n}\right)^2 + 10^{3^n} + 1 \pmod 3 &\equiv \left\{ \left(10^{3^n}\right)^2 \pmod 3 \right\} + \left\{ \left(10^{3^n} + 1\right) \pmod 3 \right\} \\ &\equiv 1 + \left\{ \left(10^{3^n} + 1\right) \pmod 3 \right\} \\ &\equiv 1 + \{(2 + \alpha_n) \pmod 3\} \\ &\equiv 1 + 2 + \{\alpha_n \pmod 3\} \\ &\equiv 0 \end{aligned}$$

where the induction hypothesis was again invoked in the last equality. □

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