PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
SPRING 2011, PROBLEM 3

WILLIAM WU

Problem A car whose wheels are of radius r feet in driven at a speed of 55 m.p.h. A particle is thrown off from one of the wheels. Neglecting air resistance, find the maximum height above the roadway which the particle can reach.

Solution Suppose the wheel rotates clockwise with speed v, and that the particle's location on the wheel upon launch is given by the angle θ, where Fig. 1 shows our orientation for θ.

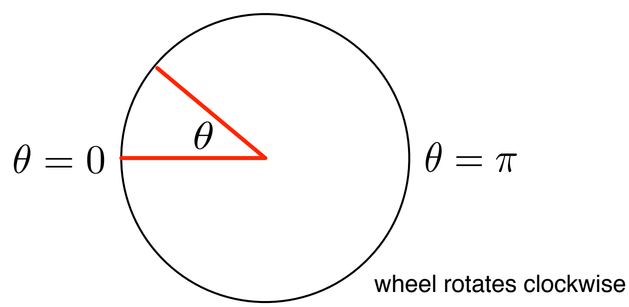

FIGURE 1. Orientation for angle θ.

Let the vertical component of the particle's velocity be denoted by v_y, where upwards is considered positive. Then $v_y = v\cos\theta$, and the initial vertical kinetic energy of the particle is $\frac{1}{2}mv_y^2$. At the peak of the particle's trajectory, all of this vertical kinetic energy is transformed into potential energy $mg\tilde{h}$, where g is the gravitational acceleration constant, and $\tilde{h}$ is the vertical distance the particle travels *relative to its initial height*. Solving for $\tilde{h}$ yields

$$\tilde{h} = \frac{v_y^2}{2g}.$$

On the other hand, the initial height of the particle is The height of the particle at its peak, relative to the ground, is $\tilde{h}$ plus $r(1+\sin\theta)$. The peak height of the particle as a function of θ is thus

$$\begin{aligned} h(\theta) &= = \tilde{h} + r(1+\sin\theta) \\ &= \frac{v^2\cos^2\theta}{2g} + r(1+\sin\theta). \end{aligned} \tag{1}$$

Date: January 31, 2011.

Differentiating with respect to θ,

$$h'(\theta) = -\frac{v^2}{g}\sin\theta\cos\theta + r\cos\theta$$

Setting $h'(\theta) = 0$, and dividing by $\cos\theta$ (this assumes that $\theta \neq 0$):

$$\begin{aligned} 0 &= -\frac{v^2}{g}\sin\theta + r \\ \implies \qquad \theta^* &= \arcsin(rg/v^2) \end{aligned}$$

Substituting θ^* into Eq. (1),

$$\begin{aligned} h(\theta^*) &= \frac{v\cos^2\theta^*}{2g} + r\left(1 + \frac{rg}{v^2}\right) \\ &= \frac{v^2}{2g}\cos^2\left(\arcsin\left(\frac{rg}{v^2}\right)\right) + r\left(1 + \frac{rg}{v^2}\right) \end{aligned}$$

By the Pythagorean theorem, $\cos\left(\arcsin\left(\frac{rg}{v^2}\right)\right) = \frac{\sqrt{v^4-(rg)^2}}{v^2}$. Note that to avoid imaginary numbers, we require $v^4 \geq (rg)^2$, or $r < v^2/g$. Hence,

$$h(\theta^*) = \begin{cases} \frac{v^2}{2g}\left(1 - \frac{r^2g^2}{v^4}\right) + r\left(1 + \frac{rg}{v^2}\right) & r < v^2/g \\ 2r & r \geq v^2/g. \end{cases} \tag{2}$$

In the latter case of $r \geq v^2/g$, the maximal height is achieved when the particle stays on the wheel and $\theta = \pi/2$. Using $v = 55 \cdot 1609.344/(3600) = 24.587$ m/s and $g = 9.83$ m/s, this transition occurs for $r = 61.49$ meters.

□

E-mail address: william.wu@themathpath.com

THE MATH PATH LLC