PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
SPRING 2011, PROBLEM 4

WILLIAM WU

Problem Show that four consecutive binomial coefficients $\binom{n}{r}, \binom{n}{r+1}, \binom{n}{r+2}, \binom{n}{r+3}$ can never be in arithmetic progression. Can you give cases of three consecutive binomial coefficients in arithmetic progression?

Solution If $C(n,r), C(n,r+1), C(n,r+2)$ are an arithmetic progression, then by definition,

$$C(n,r+1) - C(n,r) = C(n,r+2) - C(n,r+1).$$

After multiplying out the denominators and simplifying, this equation becomes

$$n^2 - 5n + 4k^2 + 8k - 4nk + 2 = 0. \tag{1}$$

Similarly, if $C(n,r+1), C(n,r+2), C(n,r+3)$ are an arithmetic progression, then

$$n^2 - 9n + 4k^2 + 16k - 4nk + 14 = 0. \tag{2}$$

Subtracting Eq. (2) from Eq. (1) yields

$$0 = 4n - 8k - 12$$

which can be rearranged as

$$n = 2k + 3. \tag{3}$$

Thus, the four binomial coefficients $C(n,r), C(n,r+1), C(n,r+2), C(n,r+3)$ are

$$\binom{2k+3}{k}, \binom{2k+3}{k+1}, \binom{2k+3}{k+2}, \binom{2k+3}{k+3}.$$

However, this cannot be an arithmetic progression since

$$\binom{2k+3}{k+1} - \binom{2k+3}{k} > 0 \qquad \text{whereas} \qquad \binom{2k+3}{k+3} - \binom{2k+3}{k+2} < 0.$$

An example of three consecutive binomial coefficients in arithmetic progression can be generated by considering solutions to Eq. (1):

$$k = \frac{1}{2}\left(-2 + n \pm \sqrt{2+n}\right).$$

We require k to be an integer. One solution is $n = 7$ and $k = 1$, yielding the three binomial coefficients $\binom{7}{1} = 7$, $\binom{7}{2} = 21$, and $\binom{7}{3} = 35$.

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E-mail address: william.wu@themathpath.com

The Math Path LLC

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