PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
SPRING 2011, PROBLEM 5

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Problem For any real numbers a, b with $a < b$, let $[a, b]$ denote the closed interval with end points a, b.

Given any finite collection of closed intervals $[a_1, b_1], \ldots, [a_n, b_n]$ such that any two of them have at least one point in common, show that there must be some point common to all the intervals.

Solution We prove by way of induction. The base case of $n = 2$ follows trivially. For the induction step, consider intervals $[a_1, b_1], \ldots, [a_n, b_n], [a_{n+1}, b_{n+1}]$, where each pair of these intervals has a non-empty intersection. By applying the induction hypothesis to the first n intervals, the intersection

$$[a, b] := \bigcap_{i=1}^{n} [a_i, b_i]$$

is non-empty. Now suppose that the proposition is false in the case of $n+1$, so that $\bigcap_{i=1}^{n+1} [a_i, b_i]$ is empty. Then

$$\begin{aligned} \emptyset &= \bigcap_{i=1}^{n+1} [a_i, b_i] \\ &= \left(\bigcap_{i=1}^{n} [a_i, b_i] \right) \cap [a_{n+1}, b_{n+1}] \\ &= [a, b] \cap [a_{n+1}, b_{n+1}] \end{aligned}$$

which says that $[a, b]$ and $[a_{n+1}, b_{n+1}]$ are disjoint. This implies that either $b_{n+1} < a$ or $b < a_{n+1}$. Without loss of generality, we only consider the former situation. Note that $a = a_i$ for some $i \in \{1, 2, \ldots, n\}$. Thus $[a_{n+1}, b_{n+1}]$ is disjoint from $[a_i, b_i]$, contradicting the premise that every pair of intervals has at least one point in common. Therefore the proposition must be true. □

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Date: February 12, 2011.