PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
SPRING 2011, PROBLEM 6

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Problem The possible scores in the game of tossing two dice are the integers $2, 3, ..., 12$. Is it possible to load the dice in such a way that these eleven scores are equally probable? Remark. Loading the dice means assigning probabilities to each of the six sides coming up. The two dice do not have to be loaded in the same way.

Solution Let a_i be the probability that the first die turns up with i pips, and let b_i be the probability that the second die turns up with i pips. Recall that

(1) the probability mass function for the sum of two independent random variables is the convolution of their individual probability mass functions, and
(2) multiplying two polynomials convolves their coefficients.

Hence, if we define the polynomials $A(x) := \sum_{i=1}^{6} a_i x^i$ and $B(x) := \sum_{i=1}^{6} b_i x^i$, then the product of these polynomials must obey the equation

$$A(x)B(x) = \sum_{i=2}^{12} \frac{1}{11} x^i$$

since we require all eleven sums $2, 3, \ldots, 12$ to be equally likely. It is possible to argue that there is no solution by showing that the system of equations created by equating coefficients is not solvable. Or alternatively, by dividing both sides by x^2,

$$\frac{A(x)}{x}\frac{B(x)}{x} = \left(\sum_{i=0}^{5} a_i x^i\right)\left(\sum_{i=0}^{5} b_i x^i\right) = \sum_{i=0}^{10}\left(\frac{1}{11}\right) x^i = \left(\frac{1}{11}\right)\left(\frac{x^{11}-1}{x-1}\right) = \left(\frac{1}{11}\right)\Phi_{11}(x).$$

The polynomial $\Phi_{11}(x)$ on the far right is the 11th cyclotomic polynomial. Since 11 is prime, $\Phi_{11}(x)$ is irreducible by Eisenstein's criterion (applied after making the substitution $x \to x+1$), and thus $\left(\frac{1}{11}\right)\Phi_{11}(x)$ has no real zeros. However, the polynomials $\frac{A(x)}{x}$ and $\frac{B(x)}{x}$ are both polynomials of odd degree, and by the intermediate value theorem they each have real zeros. Therefore $\frac{A(x)}{x}\frac{B(x)}{x} \neq \left(\frac{1}{11}\right)\Phi_{11}(x)$, and it is not possible to load the dice to make all eleven sums equally probable.

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Date: February 23, 2011.