PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
SPRING 2011, PROBLEM 7

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Problem Show that

$$\frac{1}{2\sqrt{n}} < \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} < \frac{1}{\sqrt{2n+1}}$$

for $n = 2, 3, \ldots$.

Solution We prove by way of induction. The base case of $n = 2$ can be verified by direct calculation. Now suppose the proposition is true for case n. For the case of $n+1$, the desired inequality is

$$\frac{1}{2\sqrt{n+1}} < \frac{1 \cdot 3 \cdot 5 \cdots (2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n+2)} < \frac{1}{\sqrt{2n+3}}$$

which can be rewritten as

$$\frac{1}{2\sqrt{n}} \left(\frac{\sqrt{n}}{\sqrt{n+1}} \right) < \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)} \left(\frac{2n+1}{2n+2} \right) < \frac{1}{\sqrt{2n+1}} \left(\frac{\sqrt{2n+1}}{\sqrt{2n+3}} \right).$$

By the induction hypothesis, it suffices to show that

$$\frac{\sqrt{n}}{\sqrt{n+1}} < \frac{2n+1}{2n+2} \tag{1}$$

and

$$\frac{2n+1}{2n+2} < \frac{\sqrt{2n+1}}{\sqrt{2n+3}}. \tag{2}$$

Squaring and rearranging Inequality (1),

$$\begin{aligned} \frac{n}{n+1} &\overset{?}{<} \left(\frac{2n+1}{2n+2} \right)^2 \\ n(2n+2)^2 &\overset{?}{<} (n+1)(2n+1)^2 \\ 4n^2 + 4n &\overset{?}{<} 4n^2 + 4n + 1 \\ 0 &< 1. \end{aligned}$$

Similarly, squaring and rearranging Inequality (2),

$$\begin{aligned} \left(\frac{2n+1}{2n+2} \right)^2 &\overset{?}{<} \frac{2n+1}{2n+3} \\ (2n+1)(2n+3) &\overset{?}{<} (2n+2)^2 \\ 4n^2 + 5n + 3 &\overset{?}{<} 4n^2 + 8n + 4 \\ 0 &< 3n + 1 \end{aligned}$$

which concludes the proof. □

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