PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
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Problem Find the smallest volume bounded by the coordinate planes and by a tangent plane to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \tag{1}$$

Solution Without loss of generality we may assume that $a, b, c \geq 0$. Define $F(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$; then the equation of the ellipsoid is the level curve $F(x, y, z) = 0$. Now consider a parametrized curve $x = x(t), y = y(t), z = z(t)$ residing on the ellipsoid, such that $F(x(t), y(t), z(t)) = 0$. Then by the chain rule,

$$\frac{\partial}{\partial t} F(x(t), y(t), z(t)) = \frac{\partial F}{\partial x}\frac{\partial x(t)}{\partial t} + \frac{\partial F}{\partial y}\frac{\partial y(t)}{\partial t} + \frac{\partial F}{\partial z}\frac{\partial z(t)}{\partial t} = \underbrace{\left\langle \frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \right\rangle}_{\nabla F} \cdot \left\langle \frac{\partial x(t)}{\partial t}, \frac{\partial y(t)}{\partial t}, \frac{\partial z(t)}{\partial t} \right\rangle.$$

The gradient vector ∇F is orthogonal to the tangent line at any point on the surface. Thus, the equation of the plane tangent to the ellipsoid at (x_0, y_0, z_0) is described by the plane whose normal vector is ∇F.

$$\begin{aligned} 0 &= \nabla F \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \\ &= \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \\ &= \left\langle \frac{x}{a^2}, \frac{y}{b^2}, \frac{z}{c^2} \right\rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle \\ \frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} &= \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \end{aligned}$$

Substituting Eq. (1) yields the following equation for the tangent plane:

$$\frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1. \tag{2}$$

We now compute the vertices of the tetrahedron that is bounded by this tangent plane and the coordinate axes. By setting $y = 0$ and $z = 0$, we can determine the vertex along the x-axis to be $x = a^2/x_0$. Similarly, the list of all four vertices is:

$$x\text{-vertex} : \left(\frac{a^2}{x_0}, 0, 0\right) \qquad y\text{-vertex} : \left(0, \frac{b^2}{y_0}, 0\right) \qquad z\text{-vertex} : \left(0, 0, \frac{c^2}{z_0}\right) \qquad (0,0)\text{-vertex} : (0, 0, 0)$$

The volume of this tetrahedron can be computed using the equation $V = \frac{1}{3}Ah$, where A is the area of the base and h is the height from the base to the apex. Hence,

$$V = \frac{1}{3}\left(\frac{1}{2}\frac{a^2}{x_0} \cdot \frac{b^2}{y_0}\right) \cdot \frac{c^2}{z_0} = \frac{1}{6}\frac{a^2 b^2 c^2}{x_0 y_0 z_0}.$$

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The optimization problem can now be described as follows:

$$\boxed{\begin{array}{ll} \text{minimize} & f(x,y,z) := \frac{1}{6}\frac{a^2b^2c^2}{xyz} \\ \text{subject to} & x \geq 0, y \geq 0, z \geq 0 \\ & h(x,y,z) := \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \end{array}} \tag{3}$$

over variables $x, y, z \in \mathbb{R}$. We use the method of Lagrange multipliers, ignoring the non-negativity constraints for now. From the equation $\nabla f + \lambda \nabla h = 0$,

$$\frac{1}{6}\frac{a^2b^2c^2}{x^2yz} = \lambda\frac{2x}{a^2}, \tag{4}$$

$$\frac{1}{6}\frac{a^2b^2c^2}{xy^2z} = \lambda\frac{2y}{b^2}, \tag{5}$$

$$\frac{1}{6}\frac{a^2b^2c^2}{xyz^2} = \lambda\frac{2z}{c^2}. \tag{6}$$

Multiplying both sides of the first equation by $\frac{x}{2\lambda}$, both sides of the second equation by $\frac{y}{2\lambda}$, and both sides of the third equation by $\frac{z}{2\lambda}$,

$$\frac{1}{12}\frac{a^2b^2c^2}{\lambda xyz} = \frac{x^2}{a^2}, \tag{7}$$

$$\frac{1}{12}\frac{a^2b^2c^2}{\lambda xyz} = \frac{y^2}{b^2}, \tag{8}$$

$$\frac{1}{12}\frac{a^2b^2c^2}{\lambda xyz} = \frac{z^2}{c^2}. \tag{9}$$

Summing Eqs. (7), (8), and (9), and substituting the constraint $h(x,y,z) = 1$, yields

$$\frac{1}{4}\frac{a^2b^2c^2}{\lambda xyz} = 1. \tag{10}$$

Multiplying Eq. (10) by $\frac{1}{3}$ yields

$$\frac{1}{12}\frac{a^2b^2c^2}{\lambda xyz} = 1/3. \tag{11}$$

Substituing Eq. (11) into Eqs. (7), (8), and (9) yields

$$1/3 = \frac{x^2}{a^2}, \qquad 1/3 = \frac{y^2}{b^2}, \qquad 1/3 = \frac{z^2}{c^2} \tag{12}$$

and thus

$$(x^*, y^*, z^*) = \frac{1}{\sqrt{3}}(a, b, c). \tag{13}$$

(Note that the non-negativity constraints are satisfied.) At the boundaries of the feasible set, the volume of the tetrahedron tends to infinity as the tangent plane becomes parallel to the coordinate axes. Therefore the critical point is indeed a minimum, and the minimal volume is

$$\boxed{V^* = V\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right) = \frac{\sqrt{3}}{2}abc.}$$

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