PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
SPRING 2011, PROBLEM 9

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Problem Prove that every positive integer has a multiple whose decimal representation involves the sequence 20102011.

Solution We use the pigeonhole principle. Let n be any positive integer. Consider the set S_n which contains the following $n+1$ numbers:

$$S_n := \left\{ \left(\sum_{i=0}^{k} 20102011 \cdot 10^{8i} \right) : k \in \{0, 1, 2, \ldots, n\} \right\}$$

That is, $S_n = \{20102011, 2010201120102011, 201020112010201120102011, \ldots\}$, such that the ith number is the digit sequence 20102011 repeated i times, for $i \in \{0, 1, 2, \ldots, n\}$. By the pigeonhole principle, there exist at least two distinct numbers $u, v \in S_n$ that have the same remainder modulo n:

$$u \equiv v \mod n.$$

Letting $u < v$ without loss of generality, it follows that

$$v - u \equiv 0 \mod n$$

which means there exists an integer k such that

$$nk = v - u.$$

Lastly, we observe that by the construction of the numbers in S_n, it is clear that $v - u$ must contain the digit sequence 20102011. For if $v = \sum_{i=0}^{b} 20102011 \cdot 10^{8i}$ and $u = \sum_{i=0}^{a} 20102011 \cdot 10^{8i}$, then

$$v - u = \sum_{i=a+1}^{b} 20102011 \cdot 10^{8i},$$

a number whose digits consist of 20102011 repeated $b - a$ times, followed by $8(a+1)$ zeros.

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