

PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
FALL 2012, PROBLEM 8

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Problem Let $I_k, 1 \leq k \leq n$, be intervals contained in $(0, 1)$, and suppose the sum of the lengths of these intervals is 17. Show that there is a number in $(0, 1)$ which is in at least five of the I_k .

Solution Let X be a point generated uniformly at random from $(0, 1)$, and let Y be the number of intervals that X resides in. Note that Y can be written as the sum of indicator variables $Y = \sum_{k=1}^n Y_k$, where

$$Y_k = \begin{cases} 1 & \text{if } X \in I_k \\ 0 & \text{otherwise.} \end{cases}$$

By linearity of expectation,

$$\mathbf{E}[Y] = \sum_{k=1}^n \mathbf{E}[Y_k] = \sum_{k=1}^n \mathbf{P}[X \in I_k] = \sum_{k=1}^n L_k = 17$$

where L_k is the measure of I_k . Since the expected number of intervals containing X is 17, there exists a point which belongs to at least 17 intervals, and $17 \geq 5$. \square

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