PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
FALL 2012, PROBLEM 11

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Problem If x is a positive number, and if n is a positive integer, show that

$$(1+x^{\frac{n+1}{2}})^n \leq (1+x)(1+x^2)\cdots(1+x^n).$$

Solution The function $f(u) := \log(1+x^u)$ is convex in u for any choice of positive x. This can be verified by direct differentiation: $\frac{\partial^2 f}{\partial u^2} = \frac{x^u \log(u)^2}{(1+x^u)^2} > 0$. Therefore,

$$f(\theta y_1 + (1-\theta)y_2) \leq \theta f(y_1) + (1-\theta)f(y_2)$$

for any $0 \leq \theta \leq 1$. Substituting $y_1 = i$, $y_2 = n-i+1$, and $\theta = 1/2$ yields the inequality

$$\log(1+x^{\frac{n+1}{2}}) \leq \frac{\log(1+x^i) + \log(1+x^{n-i+1})}{2}$$

which can be rearranged as

$$(1+x^{\frac{n+1}{2}})^2 \leq (1+x^i)(1+x^{n-i+1}).$$

When n is even, varying i from 1 to $n/2$ and multiplying these inequalities yields the result:

$$(1+x^{\frac{n+1}{2}})^n = \prod_{i=1}^{n/2}(1+x^{\frac{n+1}{2}})^2 \leq \prod_{i=1}^{n/2}(1+x^i)(1+x^{n-i+1}) = (1+x)(1+x^2)\cdots(1+x^n).$$

When n is odd, we may cancel $(1+x^{\frac{n+1}{2}})$ from both sides of the desired inequality and yield to the case where n is even. □

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