PURDUE UNIVERSITY MATH DEPARTMENT PROBLEM OF THE WEEK FALL 2012, PROBLEM 12

JIEHUA CHEN AND WILLIAM WU

Problem For every integer $n > 2$ let $L(n)$ denote the sum of the integers from 1 through $[n/2]$ which are relatively prime to n, and let $U(n)$ denote the sum of the integers from $[n/2]+1$ through n which are relatively prime to n. Prove that if n is divisible by 4, then $U(n)/L(n) = 3$. ([] is the greatest integer function.)

Solution Recall the identity

$$L(n) + U(n) = \sum_{1 \le k \le n, (k,n)=1} k = \phi(n)n/2,$$

where ϕ is the Euler totient function. From this identity, $L(n) = \phi([n/2])[n/2]/2$. Since n is divisible by 2, this can rewritten as

$$L(n) = \phi(n/2)n/4.$$

Euler's product formula for the totient function states that $\phi(n) = n\prod_{p|n}(1 - \frac{1}{p})$, where the product runs over distinct primes of n. Since $n|4$, $\phi(n) = 4(n/4)\prod_{p|n}(1 - \frac{1}{p})$, and $\phi(n/2) = 2(n/4)\prod_{p|n}(1 - \frac{1}{p})$, so $\phi(n/2) = \phi(n)/2$. Therefore,

$$\frac{L(n)}{L(n) + U(n)} = \frac{\phi(n/2)n/4}{\phi(n)n/2} = \frac{\phi(n)n/8}{\phi(n)n/2} = \frac{1}{4}$$

and rearranging this equation yields

$$U(n) = 3L(n).$$

□

E-mail address: jiehua.chen@themathpath.com, william.wu@themathpath.com

The Math Path, LLC

Date: November 15, 2012.