# PURDUE UNIVERSITY MATH DEPARTMENT PROBLEM OF THE WEEK SPRING 2012, PROBLEM 2

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**Problem** Find

$$\lim_{n\to\infty} \frac{1 + 2^2 + 3^3 + \ldots + (n-1)^{n-1} + n^n}{n^n}.$$

**Solution** Define $f(n) := \frac{1+2^2+3^3+\ldots+(n-1)^{n-1}+n^n}{n^n}$. We will show that $\lim_{n\to\infty} f(n) = 1$. Clearly $f(n) \geq 1$, so by the squeeze theorem, it suffices to show that $f(n)$ is upper bounded by a function $g(n)$ such that $\lim_{n\to\infty} g(n) = 1$.

$$\begin{aligned} f(n) = \frac{1}{n^n}\sum_{k=0}^{n} k^k &\leq \frac{1}{n^n}\sum_{k=0}^{n} n^k \\ &= \sum_{k=0}^{n} \frac{1}{n^{n-k}} \\ &= \sum_{k=0}^{n} \frac{1}{n^k} \\ &= \sum_{k=0}^{n} \left(\frac{1}{n}\right)^k \\ &= \frac{1-\left(\frac{1}{n}\right)^{n+1}}{1-\frac{1}{n}}. \end{aligned}$$

By the algebra of limits,

$$\lim_{n\to\infty} \frac{1-\left(\frac{1}{n}\right)^{n+1}}{1-\frac{1}{n}} = \frac{\lim_{n\to\infty} 1 - \left(\dfrac{1}{n}\right)^{n+1}}{\lim_{n\to\infty} 1 - \dfrac{1}{n}} = \frac{1}{1} = 1.$$

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