PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
SPRING 2012, PROBLEM 3

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Problem Suppose $f(x)$ is an infinitely differentiable function on (0,1) and continuous on [0,1] and satisfies $f(0) = f(1) = 0$. Prove there is an x in (0,1) such that $f(x) = f'(x)$.

Solution Let x^* denote the x such that $f(x^*) = f'(x^*)$. If $f'(0) = 0$, then $x^* = 0$, so without loss of generality we will assume that $f'(0) > 0$.

Since f is continuous and $[0, 1]$ is compact, the extreme value theorem states that f achieves a maximum value over $[0, 1]$ at some $c \in [0, 1]$. However, this c cannot equal either 0 or 1, because $f'(0) > 0$ implies by continuity that f is positive over some neighborhood of 0, whereas $f(0) = f(1) = 0$. Therefore, $c \in (0, 1)$, and $f(c) > 0$.

Lastly, define $g(x) = f'(x) - f(x)$. Then

$$g(0) = f'(0) - f(0) = f'(0)$$

and

$$g(c) = f'(c) - f(c) = -f(c)$$

where $f'(c) = 0$ because the first derivative vanishes at a local maximum. Lastly, since g is continuous, $g(0) > 0$, and $g(c) < 0$, by the intermediate value theorem there exists some $x^* \in (0, c)$ such that $g(x^*) = f'(x^*) - f(x^*) = 0$.

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