PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
SPRING 2012, PROBLEM 5

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Problem Prove that for all positive integers n the equations

$$x^2 + y^2 = 2n \tag{1}$$

$$x^2 + y^2 = n \tag{2}$$

have the same number of integer solutions.

Solution Suppose $(x, y) = (a, b)$ is an integer solution to $x^2 + y^2 = n$. Then $(x, y) = (a - b, a + b)$ is an integer solution to $x^2 + y^2 = 2n$, since

$$\begin{aligned} 2n &= 2|a + bi|^2 \\ &= |1 + i|^2|a + bi|^2 \\ &= |(1 + i)(a + bi)|^2 \\ &= |(a - b) + (a + b)i|^2 \\ &= (a - b)^2 + (a + b)^2. \end{aligned}$$

Conversely, we now show that if $(x, y) = (c, d)$ is an integer solution to $x^2 + y^2 = 2n$, then $(x, y) = (\frac{c+d}{2}, \frac{c-d}{2})$ is an integer solution to $x^2 + y^2 = n$. First we verify that $\frac{c+d}{2}$ and $\frac{c-d}{2}$ are integers. Computing the remainder of both sides of Eq. (1) modulo 2,

$$\begin{aligned} \left(c^2 + d^2\right) \bmod 2 &= (2n) \bmod 2 \\ (c^2 \bmod 2) + (d^2 \bmod 2) &= 0. \end{aligned}$$

Recall that the square of an even number is even, and the square of an odd number is odd. Therefore, c and d must have the same parity for the above equation to hold. Consequently, $c + d$ and $c - d$ are both even, and thus $\frac{c+d}{2}$ and $\frac{c-d}{2}$ are integers. Lastly, we verify that these fractions are solutions to Eq. (2) by direct substitution:

$$\begin{aligned} \left(\frac{c + d}{2}\right)^2 + \left(\frac{c - d}{2}\right)^2 &= \frac{c^2 + 2cd + d^2}{4} + \frac{c^2 - 2cd + d^2}{4} \\ &= \frac{2c^2 + 2d^2}{4} \\ &= \frac{c^2 + d^2}{2} \\ &= \frac{2n}{2} \\ &= n. \end{aligned}$$

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