PURDUE UNIV. MATH DEPT.: PROBLEM OF THE WEEK
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Problem An urn has four balls numbered 1, 2, 3, 4. They are drawn one at a time at random with replacement, that is, a ball is drawn, its number is noted, and the ball is replaced and the urn is mixed before the next draw. The draws continue until a number is drawn that is smaller than a previously drawn number. Find the probability that the last number drawn is 1.

Solution A drawn number is smaller than a previously drawn number if and only if it is smaller than the largest number drawn so far. Therefore, to analyze the stopping condition, we only need to remember the largest number seen so far, which evolves as a Markov process.

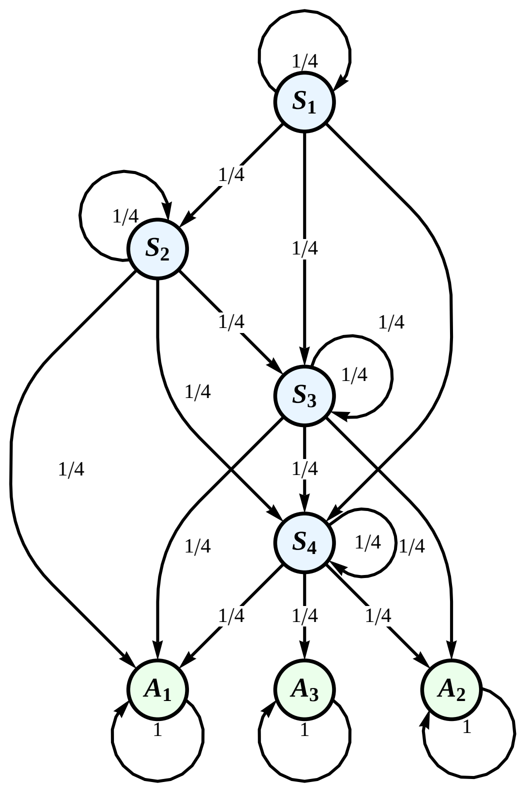

FIGURE 1. Markov chain for largest number seen so far, with absorbing states.

Consider the Markov chain illustrated in Fig. (1). It has seven states, four of which are transient states labeled S_1,S_2,S_3,S_4, and three of which are absorbing states labeled A_1,A_2,A_3. At each time step, we draw a number i with probability $1/4$. If i is the largest number drawn so far, move to S_i. Otherwise, i is smaller than the largest number so far, and we move to absorbing state A_i, where the process terminates. The process starts in S_1, because we initially set the largest number seen so far to the smallest possible number. The probability transition

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matrix for this Markov chain is

$$\mathbf{P} = \left(\begin{array}{c||cccc|ccc} & S_1 & S_2 & S_3 & S_4 & A_1 & A_2 & A_3 \\ \hline S_1 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 & 0 \\ S_2 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 & 0 \\ S_3 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 & 0 \\ S_4 & 0 & 0 & 0 & 1/4 & 1/4 & 1/4 & 1/4 \\ \hline A_1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ A_2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) := \left(\begin{array}{c|c} \mathbf{Q} & \mathbf{R} \\ \hline 0 & \mathbf{I} \end{array}\right)$$

where $\mathbf{P}_{u,v}$ is the probability of transitioning from state u to state v. Now consider the 4-by-3 transient-to-absorbing matrix $\mathbf{U}$, in which $\mathbf{U}_{u,v}$ denotes the probability of starting in transient state $u \in \{S_1, S_2, S_3, S_4\}$ and ultimately being absorbed in state $v \in \{A_1, A_2, A_3\}$. By first step analysis, $\mathbf{U}$ satisfies the recursion

$$\mathbf{U} = \mathbf{QU} + \mathbf{R}$$

and thus

$$\mathbf{U} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R}$$

where $(\mathbf{I} - \mathbf{Q})^{-1}$ is known as the fundamental matrix and is well-defined. A direct matrix computation yields

$$\mathbf{U} = \left(\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & \frac{1}{4} & \frac{1}{4} \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \right)^{-1} \begin{pmatrix} 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 \\ \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{37}{81} & \frac{28}{81} & \frac{16}{81} \\ \frac{16}{27} & \frac{7}{27} & \frac{4}{27} \\ \frac{4}{9} & \frac{4}{9} & \frac{1}{9} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix}.$$

Therefore, the probability of starting in S_1 and ending in A_1 is $\mathbf{U}_{S_1,A_1} = 37/81 \approxeq 0.45679$. This can also be empirically verified with a simple Monte Carlo simulation:

Listing 1. Monte Carlo Simulation of Absorption Probability

```c
#include <stdio.h>
#include <stdlib.h>
int main(void) {
  int n = 100000000;
  int tally[] = {0,0,0,0};
  int k, i, max_so_far;
  for (k=0;k<n;k++) {
    max_so_far = 0;
    while (1) {
      i = rand() % 4;
      if (i >= max_so_far) max_so_far = i;
      if (i < max_so_far) {
        tally[i]++;
        break;
      }
    }
  }
  printf ("Probability: %lf.\n",(double)tally[0]/n);
  return 0;
}
```

After 100,000,000 trials, the program output is: `Probability:  0.456790.`

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