PURDUE UNIVERSITY MATH DEPARTMENT
PROBLEM OF THE WEEK
SPRING 2013, PROBLEM 2

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Problem A random walk on the three dimensional integer lattice is dened as follows. The walker starts at $(0, 0, 0)$. A standard six sided die is rolled six times. After each roll the walker moves to one of its six nearest neighbors, according to the following protocol: if the die rolls 1, 2, 3, 4, 5, or 6 dots the walker jumps one unit in the $+x, -x, +y, -y, z, -z$ direction respectively. Find the probability that after the sixth roll the walker is back at its starting point $(0, 0, 0)$.

Solution For any random walk that returns to the origin, each of the three directions x, y, and z must have an equal number of positive and negative steps. From this observation it is straightforward to enumerate the following types of random walks, where ordering is ignored:

(1) CATEGORY I

- $x : +-$
- $y : +-$
- $z : +-$

(2) CATEGORY II

- $x : ++--$
- $y : +-$
- $z :$

- $x : +-$
- $y : ++--$
- $z :$

- $x : +-$
- $y :$
- $z : ++--$

- $x : ++--$
- $y :$
- $z : +-$

- $x :$
- $y : ++--$
- $z : +-$

- $x :$
- $y : +-$
- $z : ++--$

(3) CATEGORY III

- $x : +++---$
- $y :$
- $z :$

- $x :$
- $y : +++---$
- $z :$

- $x :$
- $y :$
- $z : +++---$

Accounting for orderings, the number of sequences in Category I is 6!, since all six items in the sequence are distinguishable. In Category II, there are six subcategories, and in each

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subcategory the number of sequences is $6!/(2!2!)$, since there are two pairs of indistinguishable items. In Category II, there are three subcategories, and in each subcategory the number of sequences is $6!/(3!3!)$, since there are two triplets of indistinguishable items. Consequently, the probability of returning to the origin is:

$$\frac{6! + 6 \cdot \frac{6!}{2!2!} + 3\frac{6!}{3!3!}}{6^6} = 155/3888 \approxeq 0.039866.$$

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