WORKED EXAMPLES OF QUANTITATIVE FINANCE ANALYSES

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1. Straddle Pricing and Replicating Strategy

A straddle is an option with payoff $|S_T - K|$ at maturity T.

(a) Under the Black-Scholes model, what is the price at time $t \le T$ of this option?

Solution In the following, $a := (r - \sigma^2/2)(T-t)$, $b := \sigma\sqrt{T-t}$, and $Z \sim \mathcal{N}(0,1)$.

$$\begin{aligned} C(t, S_t) &= e^{-r(T-t)}\mathbf{E}^{\mathbb{Q}}\left[|S \exp a + bZ - K|\right] \\ &= e^{-r(T-t)}\left(\frac{1}{\sqrt{2\pi}}\int_{-d_-}^{\infty}(Se^{a+bz} - K)e^{-z^2/2}dz - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d_-}(Se^{a+bz} - K)e^{-z^2/2}dz\right)\end{aligned}$$

where $-d_-$ is the value of z such that $Se^{a+bz} - K = 0$, and we know the negative sign must be applied to the second integral since $b = \sigma\sqrt{T-t} > 0$.

$$\begin{aligned} C(t, S_t) = e^{-r(T-t)}&\left(\frac{1}{\sqrt{2\pi}}\int_{-d_-}^{\infty} Se^{a+bz}e^{-z^2/2}dz - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d_-} Se^{a+bz}e^{-z^2/2}dz\right. \\ &\quad\left. + \frac{1}{\sqrt{2\pi}}\int_{-d_-}^{\infty} -Ke^{-z^2/2}dz + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d_-} Ke^{-z^2/2}dz\right) \\ = e^{-r(T-t)}&\left(\frac{1}{\sqrt{2\pi}}\int_{-d_-}^{\infty} Se^{a+bz}e^{-z^2/2}dz - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d_-} Se^{a+bz}e^{-z^2/2}dz + K(1 - 2\mathcal{N}(-d_-))\right) \\ = e^{-r(T-t)}&\left(Se^{a+\frac{1}{2}b^2}\mathcal{N}(d_- + b) + K(1 - 2\mathcal{N}(-d_-)) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{-d_-} Se^{a+bz}e^{-z^2/2}dz\right) \\ = e^{-r(T-t)}&\left(Se^{a+\frac{1}{2}b^2}\mathcal{N}(d_- + b) + K(1 - 2\mathcal{N}(-d_-)) + \frac{1}{\sqrt{2\pi}}\int_{\infty}^{d_-} Se^{a-bu}e^{-u^2/2}du\right) \\ = e^{-r(T-t)}&\left(Se^{a+\frac{1}{2}b^2}\mathcal{N}(d_+) + K(1 - 2\mathcal{N}(-d_-)) - \frac{1}{\sqrt{2\pi}}\int_{d_-}^{\infty} Se^{a-bu}e^{-u^2/2}du\right)\end{aligned}$$

where we define $d_+ = d_- + b = \frac{\log\frac{S}{K} + (r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$. To carry out the last integrand:

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}}\int_{d_-}^{\infty} Se^{a-bu}e^{-u^2/2}du &= Se^a\frac{1}{\sqrt{2\pi}}\int_{d_-}^{\infty} e^{-bu}e^{-u^2/2}du \\
&= Se^a\frac{1}{\sqrt{2\pi}}\int_{d_-}^{\infty} e^{-\frac{1}{2}(u^2+2bu)}du \\
&= Se^a\frac{1}{\sqrt{2\pi}}\int_{d_-}^{\infty} e^{-\frac{1}{2}(u^2+2bu+b^2-b^2)}du \\
&= Se^{a+\frac{1}{2}b^2}\frac{1}{\sqrt{2\pi}}\int_{d_-}^{\infty} e^{-\frac{1}{2}(u+b)^2}du \\
&= Se^{a+\frac{1}{2}b^2}\frac{1}{\sqrt{2\pi}}\int_{d_-+b}^{\infty} e^{-\frac{1}{2}v^2}dv \\
&= Se^{a+\frac{1}{2}b^2}(1-\mathcal{N}(d_-+b)) \\
&= Se^{a+\frac{1}{2}b^2}(1-\mathcal{N}(d_+)).
\end{aligned}$$

Combining the results:

$$C(t,S_t) = e^{-r(T-t)}\left(Se^{a+\frac{1}{2}b^2}\mathcal{N}(d_+) + K(1-2\mathcal{N}(-d_-)) - Se^{a+\frac{1}{2}b^2}(1-\mathcal{N}(d_+))\right).$$

Since $a + \frac{1}{2}b^2 = r(T-t)$, we can distribute the $e^{-r(T-t)}$ to get

$$C(t,S_t) = S\mathcal{N}(d_+) + Ke^{-r(T-t)}(1-2\mathcal{N}(-d_-)) - S(1-\mathcal{N}(d_+))$$

Rearranging, and letting $\tau := T-t$, yields

$$\begin{aligned}
C(t,S_t) &= S(2\mathcal{N}(d_+)-1) + Ke^{-r\tau}(1-2\mathcal{N}(-d_-)) \\
d_+ &= \frac{\log\frac{S}{K} + (r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}} \\
d_- &= \frac{\log\frac{S}{K} + (r-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}.
\end{aligned}$$

□

(b) Under the Black-Scholes model, what is the replicating strategy?

Solution The quantity of stock to hold at time t is $\phi_t = \frac{\partial}{\partial C}S$, and the quantity of money in the portfolio at time t is $C(t,S) - \frac{\partial}{\partial C}S$, where

$$\begin{aligned}
\frac{\partial}{\partial C}S &= -1 + 2\mathcal{N}(d_+) + \frac{2e^{-\frac{\left(2r+\sigma^2\right)^2\tau^2+4\mathrm{Log}\left[\frac{S}{K}\right]^2}{8\sigma^2\tau}}\sqrt{\frac{2}{\pi}}\left(\frac{S}{K}\right)^{-\frac{1}{2}-\frac{r}{\sigma^2}}}{\sigma\sqrt{\tau}}, \\
d_+ &= \frac{\log\frac{S}{K} + (r+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}.
\end{aligned}$$

□

2. Black-Scholes Algebra

The Black-Scholes pricing formula for a European call option is given by

$$C(S,t) = SN(d_1) - E\exp(-r(T-t))N(d_2)$$

Show that

(1) $$S\exp(-d_1^2/2) = E\exp(-r(T-t))\exp(-d_2^2/2).$$

Solution If we divide both sides of Eq. (1) by S, then the problem is equivalent to showing that

$$\exp(-d_1^2/2) = E/S \exp(-r(T-t)) \exp(-d_2^2/2). \tag{2}$$

Based on the definitions, the right-hand side (RHS) of Eq. (2) is

$$\begin{aligned}
RHS &= \exp\left(\log(E/S) - r(T-t) - \frac{(\log(S/E) + (r - 1/2\sigma^2)(T-t))^2}{2\sigma^2(T-t)}\right) \\
&= \exp\left(\frac{2\sigma^2(T-t)\log(E/S) - 2\sigma^2 r(T-t)^2}{2\sigma^2(T-t)}\right) \\
&\quad \times \exp\left(\frac{-\log^2(S/E) - 2\log(S/E)(r-1/2\sigma^2)(T-t) - (r-1/2\sigma^2)^2(T-t)^2}{2\sigma^2(T-t)}\right) \\
&= \exp\left(\frac{-\log^2(S/E)}{2\sigma^2(T-t)}\right) \\
&\quad \times \exp\left(\frac{-2\sigma^2(T-t)\log(S/E) - 2(T-t)\log(S/E)(r-1/2\sigma^2)}{2\sigma^2(T-t)}\right) \\
&\quad \times \exp\left(\frac{-2\sigma^2 r(T-t)^2 - (r-1/2\sigma^2)^2(T-t)^2}{2\sigma^2(T-t)}\right) \\
&= \exp\left(\frac{-\log^2(S/E)}{2\sigma^2(T-t)}\right) \\
&\quad \times \exp\left(\frac{-2(T-t)r\log(S/E) + \sigma^2(T-t)\log(S/E) - 2\sigma^2(T-t)\log(S/E)}{2\sigma^2(T-t)}\right) \\
&\quad \times \exp\left(\frac{-r^2(T-t)^2 + r\sigma^2(T-t)^2 - (1/2\sigma^2)^2(T-t)^2 - 2r\sigma^2(T-t)^2}{2\sigma^2(T-t)}\right) \\
&= \exp\left(\frac{-\log^2(S/E)}{2\sigma^2(T-t)}\right) \\
&\quad \times \exp\left(\frac{-2(T-t)r\log(S/E) - \sigma^2(T-t)\log(S/E)}{2\sigma^2(T-t)}\right) \\
&\quad \times \exp\left(\frac{-r^2(T-t)^2 - r\sigma^2(T-t)^2 - (1/2\sigma^2)^2(T-t)^2}{2\sigma^2(T-t)}\right) \\
&= \exp\left(\frac{-\log^2(S/E) - 2(T-t)(r+\sigma^2/2)\log(S/E)}{2\sigma^2(T-t)}\right) \\
&\quad \times \exp\left(\frac{-(r+\sigma^2/2)^2(T-t)^2}{2\sigma^2(T-t)}\right) \\
&= \exp\left(\frac{-\log^2(S/E) - 2(T-t)(r+\sigma^2/2)\log(S/E) - (r+1/2\sigma^2)^2(T-t)^2}{2\sigma^2(T-t)}\right) \\
&= \exp\left(-\frac{(\log(S/E) + (T-t)(r+\sigma^2/2))^2}{2\sigma^2(T-t)}\right) \\
&= \exp(-1/2d_1^2)
\end{aligned}$$

which equals the left-hand side of Eq. (2).

□

3. Feynman-Kac Derivations

(a) Derivation of Feynman-Kac Formula:

Show that if the function $W(t,s)$ solves the boundary value problem

$$\frac{\partial W}{\partial t}(t,s)+\mu(t,s)\frac{\partial W}{\partial s}(t,s)+\frac{1}{2}\sigma^2(t,s)\frac{\partial^2 W}{\partial s^2}(t,s)=0 \qquad (3)$$
$$W(T,s)=G(s)$$

and the process $S(t)$ obeys the dynamics

$$dS_t=\mu(t,S_t)dt+\sigma(t,S_t)dX(t)$$

where $X(t)$ is a Brownian motion, then W can be represented as

$$W(t,S_t)=\mathbf{E}\left[G(S_T)|\mathcal{F}_t\right]$$

where $\mathcal{F}_t$ is the filtration up to time t, and $S(t)$ and $W(t,S_t)$ are $\mathcal{F}_t$-adapted.

Solution We first apply the Ito formula to derive an integral equation for some function $W(t,S_t)$ over $[t,T]$:

$$\begin{aligned}
dW(S_t,t) &= \frac{\partial W(S_t,t)}{\partial S_t}dS_t+\frac{\partial W(S_t,t)}{\partial t}dt+1/2\frac{\partial^2 W(S_t,t)}{\partial {S_t}^2}dS_tdS_t\\
&= \frac{\partial W(S_t,t)}{\partial t}dt+\frac{\partial W(S_t,t)}{\partial S_t}(\mu S_tdt+\sigma dX(t))+1/2\frac{\partial^2 W(S_t,t)}{\partial {S_t}^2}\sigma^2S_t^2dt\\
&= \left(\frac{\partial W(S_t,t)}{\partial t}+\mu\frac{\partial W(S_t,t)}{\partial S_t}S_t+1/2\frac{\partial^2 W(S_t,t)}{\partial {S_t}^2}\sigma^2S_t^2\right)dt+\sigma S_t\frac{\partial W(S_t,t)}{\partial S_t}dX(t)\\
&= \sigma S_t\frac{\partial W(S_t,t)}{\partial S_t}dX(t).
\end{aligned}$$

Integrating both sides of this SDE,

$$W(S_T,T) = W(S_t,t)+\int_t^T\sigma S_t\frac{\partial W(S_t,t)}{\partial S_t}dX(t).$$

Now applying the terminal condition $W(T,s)=G(s)$,

$$G(S_T) = W(S_t,t)+\int_t^T\sigma S_t\frac{\partial W(S_t,t)}{\partial S_t}dX(t).$$

Taking the conditional expectation of both sides with respect to $\mathcal{F}_t$,

$$\begin{aligned}
\mathbf{E}\left[G(S_T)\mid\mathcal{F}_t\right] &= W(S_t,t)+\underbrace{\mathbf{E}\left[\int_t^T\sigma S_t\frac{\partial W(S_t,t)}{\partial S_t}dX(t)\mid\mathcal{F}_t\right]}_{=0}\\
&= W(S_t,t).
\end{aligned}$$

□

(b) Derive the boundary value problem associated with the expectation

$$V(t,S_t)=\exp(-r(T-t))\mathbf{E}\left[G(S_T)|\mathcal{F}_t\right]$$

where r is a positive constant.

Solution When $V(t,s)=\exp(-r(T-t))W(t,s)$,

$$\begin{aligned}
&\frac{\partial V(S_t,t)}{\partial t}+\mu\frac{\partial V(S_t,t)}{\partial S_t}S_t+\frac{1}{2}\frac{\partial^2 V(S_t,t)}{\partial {S_t}^2}\sigma^2S_t^2\\
&=\exp(-r(T-t))\underbrace{\left(\frac{\partial W(S_t,t)}{\partial t}+\mu\frac{\partial W(S_t,t)}{\partial S_t}S_t+\frac{1}{2}\frac{\partial^2 W(S_t,t)}{\partial {S_t}^2}\sigma^2S_t^2\right)}_{=0}+rV(t,s)\\
&=rV(t,s).
\end{aligned}$$

□

4. Coupon Bond Discrete Compounding Analysis

A coupon bond pays out 3% each year on a principal of 100. The bond matures in 5 years and is currently priced at 90. Find the yield to maturity, duration and convexity of the bond.

Solution

- Yield to maturity:
  Let y denote the yield to maturity. Then

$$V = P\exp(-y(T-t)) + \sum_{i=1}^{N} C_i \exp(-y(t_i - t))$$

where $V = 90, P = 100, t = 0, T = 5$, and $C_i = 3$ for each i. Substituting these parameters,

$$90 = 100\exp(-5\cdot y) + \sum_{k=1}^{5} 3\exp(-y\cdot k)$$

This is a fifth-order polynomial in $\exp(y)$. Using a numerical solver, we can verify that it has only one real root. Taking the logarithm of this real root yields $y = 0.0519$.

- Duration:

$$-\frac{1}{V}\frac{dV}{dy} = \frac{1}{V}\left[P(T-t)\exp(-y(T-t)) + \sum_{i=1}^{N} C_i(t_i - t)\exp(-y(t_i - t))\right],$$

Substituting $y = 0.0519$, the duration is equal to 4.70.

- Convexity:

$$\frac{1}{V}\frac{d^2V}{d^2y} = \frac{1}{V}\left[P(T-t)^2\exp(-y(T-t)) + \sum_{i=1}^{N} C_i(t_i - t)^2\exp(-y(t_i - t))\right],$$

Substituting $y = 0.0519$, the convexity is equal to 22.91.

□

5. Pricing of Up-and-Out Barrier Option

Suppose that (S_t) satisfies the Black-Scholes SDE. We denote W the Brownian motion under $\mathbb{Q}$. Our goal is to compute the price of an up-and-out barrier option of the binary type, that is a contract which pays 1 dollar at time T if the stock price never was above B on $[0, T]$, that is a contract with payoff at time T equal to

$$H_T = \mathbf{1}[\sup_{0\le t\le T} S_t < B].$$

(a) Show that the price at time 0 of this option is of the form

$$e^{-rT}\mathbb{Q}\{\sup_{0\le s\le T} W_s + \nu s < b\}.$$

Solution Let $G(T)$ denote the price of the up-and-out barrier option.

$$\begin{aligned}
G(S_0) &= \mathbf{E}^{\mathbb{Q}}\left[e^{-rT}\mathbf{1}[\sup_{0\le t\le T} S_t < B]\right] \\
&= e^{-rT}\mathbf{E}^{\mathbb{Q}}\left[\mathbf{1}[\sup_{0\le t\le T} S_t < B]\right] \\
&= e^{-rT}\mathbb{Q}\left\{\sup_{0\le t\le T} S_t < B\right\} \\
&= e^{-rT}\mathbb{Q}\left\{\sup_{0\le t\le T} S_0\exp{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t} < B\right\} \\
&= e^{-rT}\mathbb{Q}\left\{\sup_{0\le t\le T} \exp{(\mu-\frac{1}{2}\sigma^2)t+\sigma W_t} < \frac{B}{S_0}\right\} \\
&= e^{-rT}\mathbb{Q}\left\{\sup_{0\le t\le T} (\mu-\frac{1}{2}\sigma^2)t+\sigma W_t < \log\frac{B}{S_0}\right\} \\
&= e^{-rT}\mathbb{Q}\left\{\sup_{0\le t\le T} W_t + \frac{\mu-\frac{1}{2}\sigma^2}{\sigma}t < \frac{1}{\sigma}\log\frac{B}{S_0}\right\} \\
&= e^{-rT}\mathbb{Q}\left\{\sup_{0\le t\le T} W_t + \nu t < b\right\}
\end{aligned}$$

where $\nu = \frac{\mu-\frac{1}{2}\sigma^2}{\sigma}$ and $\frac{1}{\sigma}\log\frac{B}{S_0}$. □

(b) Let ν be a real number and define a probability measure $\hat{\mathbb{Q}}$ by

$$\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}}|_{\mathcal{F}_t} = e^{-\nu^2 t/2-\nu W_t}.$$

Girsanov tells us that if we let $\hat{W}_s = W_s + \nu s$, then $(\hat{W}_s)$ is a Brownian motion under $\hat{\mathbb{Q}}$. Show that for any bounded function f, we have

$$\mathbf{E}^{\hat{\mathbb{Q}}}\left[f(\hat{W}_t, \sup_{0\le s\le t}\hat{W}_s)e^{-\nu^2 t/2+\nu\hat{W}_t}\right] = \mathbf{E}^{\mathbb{Q}}\left[f(W_t+\nu t, \sup_{0\le s\le t}(W_s+\nu s))\right].$$

Solution We use the following notation: let $X_t^* := \sup_{0\le s\le t} X_t$. Then,

$$\begin{aligned}
\mathbf{E}^{\hat{\mathbb{Q}}}\left[f\left(\hat{W}_t, \sup_{0\le s\le t}\hat{W}_s\right)e^{-\nu^2 t/2+\nu\hat{W}_t}\right] &= \mathbf{E}^{\hat{\mathbb{Q}}}\left[f(\hat{W}_t,\hat{W}_t^*)e^{-\nu^2 t/2+\nu\hat{W}_t}\right] \\
&= \int f(\hat{W}_t(\omega),\hat{W}_t^*(\omega))e^{-\nu^2 t/2+\nu\hat{W}_t(\omega)}d\hat{\mathbb{Q}}(\omega) \\
&= \int f(\hat{W}_t(\omega),\hat{W}_t^*(\omega))e^{-\nu^2 t/2+\nu\hat{W}_t(\omega)}\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}}(\omega)d\mathbb{Q}(\omega)
\end{aligned}$$

We apply the substitutions $\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}}|_{\mathcal{F}_t} = e^{-\nu^2 t/2-\nu W_t}$ and $\hat{W}_s = W_s+\nu s$:

$$\begin{aligned}
\mathbf{E}^{\hat{\mathbb{Q}}}\left[f(\hat{W}_t,\hat{W}_t^*)e^{-\nu^2 t/2+\nu\hat{W}_t}\right] &= \int f(\hat{W}_t(\omega),\hat{W}_t^*(\omega))e^{-\nu^2 t/2+\nu\hat{W}_t(\omega)}\frac{d\hat{\mathbb{Q}}}{d\mathbb{Q}}(\omega)d\mathbb{Q}(\omega) \\
&= \int f(W_t(\omega)+\nu t, \left(\sup_{0\le s\le t}(W_s+\nu s)\right)(\omega))e^{-\nu^2 t/2+\nu(W_t(\omega)+\nu t)}e^{-\nu^2 t/2-\nu W_t(\omega)}d\mathbb{Q}(\omega)
\end{aligned}$$

The exponentials simplify:

$$-\nu^2 t/2 + \nu(W_t(\omega) + \nu t) - \nu^2 t/2 - \nu W_t(\omega) = 0$$

and thus

$$\begin{aligned}\mathbf{E}^{\hat{\mathbb{Q}}}\left[f(\hat{W}_t, \hat{W}_t^*)e^{-\nu^2 t/2+\nu\hat{W}_t}\right] &= \int f(W_t(\omega) + \nu t, \left(\sup_{0\le s\le t}(W_s + \nu s)\right)(\omega))d\mathbb{Q}(\omega)\\ &= \mathbf{E}^{\mathbb{Q}}\left[f(W_t + \nu t, \sup_{0\le s\le t}(W_s + \nu t)\right].\end{aligned}$$

□

(c) Find an expression for $\mathbf{E}^{\mathbb{Q}}\left[f(W_t + \nu t, \sup_{0\le s\le t}(W_s + \nu s)\right]$, given that

$$\mathbb{Q}\left\{W_t \in da, \sup_{0\le s\le t} W_s \in db\right\} = \frac{2(2b-a)}{\sqrt{2\pi t^3}}\exp -\frac{(2b-a)^2}{2t}\mathbf{1}[b\ge 0]\mathbf{1}[a\le b]dadb.$$

Solution From the result of part (b),

$$\mathbf{E}^{\mathbb{Q}}\left[f(W_t + \nu t, \sup_{0\le s\le t}(W_s + \nu s)\right] = \mathbf{E}^{\hat{\mathbb{Q}}}\left[f(\hat{W}_t, \sup_{0\le s\le t}\hat{W}_s)e^{-\nu^2 t/2+\nu\hat{W}_t}\right].$$

The right-hand side expectation can be expanded as

$$\int_{\mathbb{R}^2}\left(\text{joint density of } \hat{W}_t \text{ and } \sup_{0\le s\le t}\hat{W}_t\right)\left(\text{the function } f(\hat{W}_t, \sup_{0\le s\le t}\hat{W}_s)e^{-\nu^2 t/2+\nu\hat{W}_t}, \text{ evaluated over } \mathbb{R}^2\right).$$

Letting a denote a possible value of $\hat{W}_t$, and letting b denote a possible value of $\sup_{0\le s\le t}\hat{W}_s$, we have

$$\begin{aligned}\mathbf{E}^{\hat{\mathbb{Q}}}\left[f(\hat{W}_t, \sup_{0\le s\le t}\hat{W}_s)e^{-\nu^2 t/2+\nu\hat{W}_t}\right] &= \int_{\mathbb{R}^2}\left(\frac{2(2b-a)}{\sqrt{2\pi t^3}}\exp -\frac{(2b-a)^2}{2t}\mathbf{1}[b\ge 0]\mathbf{1}[a\le b]\right)\left(f(a,b)e^{-\nu^2 t/2+\nu a}\right)dadb\\ &= \int_{\mathbb{R}^2} f(a,b)\frac{2(2b-a)}{\sqrt{2\pi t^3}}\exp -\frac{(2b-a)^2}{2t} - \nu^2 t/2 + \nu a\mathbf{1}[b\ge 0]\mathbf{1}[a\le b]dadb\\ &= \int_{\mathbb{R}^2} f(a,b)\frac{2(2b-a)}{\sqrt{2\pi t^3}}\exp 2b\nu - \frac{(2b-a+\nu t)^2}{2t}\mathbf{1}[b\ge 0]\mathbf{1}[a\le b]dadb\end{aligned}$$

where the last equality can be verified by expanding the polynomials. □

(d) Deduce that when $b \ge 0$ and $a \le b$,

$$\mathbb{Q}\left\{\sup_{0\le s\le t}(W_s + \nu s) \ge b | W_t + \nu t = a\right\} = e^{2b(a-b)/2}$$

then compute the value of

$$\mathbb{Q}\left\{\sup_{0\le s\le t}(W_s + \nu s) \ge b\right\}.$$

Solution Let $f(A,B) = \mathbf{1}[A = \tilde{a}, B \geq \tilde{b}]$, for specific values $\tilde{a}$ and $\tilde{b}$ satisfying $\tilde{a} \leq \tilde{b}$ and $\tilde{b} \geq 0$. f is a bounded function, so using the result of part (b), we have

$$\begin{aligned}
\mathbb{Q}\left\{W_t + \nu t = \tilde{a}, \sup_{0\leq s\leq t}(W_s + \nu s) \geq \tilde{b}\right\} &= \mathbf{E}^{\mathbb{Q}}\left[\mathbf{1}[W_t + \nu t = \tilde{a}, \sup_{0\leq s\leq t}(W_s + \nu s) \geq \tilde{b}]\right] \\
&= \mathbf{E}^{\mathbb{Q}}\left[f(W_t + \nu t, \sup_{0\leq s\leq t}(W_s + \nu s)\right] \\
&= \mathbf{E}^{\mathbb{Q}}\left[f(\hat{W}_t, \sup_{0\leq s\leq t}\hat{W}_s)e^{-\nu^2 t/2 + \nu\hat{W}_t}\right] \\
&= \int_{\mathbb{R}^2} f(a,b)\frac{2(2b-a)}{\sqrt{2\pi t^3}}\exp -\frac{(2b-a)^2}{2t} - \frac{\nu^2 t}{2} + \nu a\mathbf{1}[b \geq 0]\mathbf{1}[a \leq b]dadb \\
&= \int_{b=\tilde{b}}^{\infty}\frac{2(2b-\tilde{a})}{\sqrt{2\pi t^3}}\exp -\frac{(2b-\tilde{a})^2}{2t} - \frac{\nu^2 t}{2} + \nu\tilde{a}db \\
&= \frac{2}{\sqrt{2\pi t^3}}\exp -\frac{\nu^2 t}{2} + \nu\tilde{a}\int_{b=\tilde{b}}^{\infty}(2b-\tilde{a})\exp -\frac{(2b-\tilde{a})^2}{2t}db
\end{aligned}$$

To evaluate the integral, set $u = 2b - \tilde{a}$. Then the limits of integration become $2\tilde{b} - \tilde{a}$ and ∞, and $db = \frac{1}{2}du$. Consequently,

$$\begin{aligned}
\mathbb{Q}\left\{W_t + \nu t = \tilde{a}, \sup_{0\leq s\leq t}(W_s + \nu s) \geq \tilde{b}\right\} &= \frac{2}{\sqrt{2\pi t^3}}\exp -\frac{\nu^2 t}{2} + \nu\tilde{a}\int_{b=\tilde{b}}^{\infty}(2b-\tilde{a})\exp -\frac{(2b-\tilde{a})^2}{2t}db \\
&= \frac{2}{\sqrt{2\pi t^3}}\exp -\frac{\nu^2 t}{2} + \nu\tilde{a}\int_{u=2\tilde{b}-\tilde{a}}^{\infty} u\exp -\frac{u^2}{2t}\frac{1}{2}du \\
&= \frac{1}{\sqrt{2\pi t^3}}\exp -\frac{\nu^2 t}{2} + \nu\tilde{a}\int_{u=2\tilde{b}-\tilde{a}}^{\infty} u\exp -\frac{u^2}{2t}du \\
&= \frac{-t}{\sqrt{2\pi t^3}}\exp -\frac{\nu^2 t}{2} + \nu\tilde{a}\int_{u=2\tilde{b}-\tilde{a}}^{\infty}\left(-\frac{1}{t}\right)u\exp -\frac{u^2}{2t}du \\
&= \frac{-t}{\sqrt{2\pi t^3}}\exp -\frac{\nu^2 t}{2} + \nu\tilde{a}\int_{u=2\tilde{b}-\tilde{a}}^{\infty}\left(-\frac{u}{t}\right)\exp -\frac{u^2}{2t}du \\
&= \frac{-t}{\sqrt{2\pi t^3}}\exp -\frac{\nu^2 t}{2} + \nu\tilde{a}\exp -\frac{u^2}{2t}\Bigg]_{2\tilde{b}-\tilde{a}}^{\infty} \\
&= \frac{-t}{\sqrt{2\pi t^3}}\exp -\frac{\nu^2 t}{2} + \nu\tilde{a}\left(-\exp -\frac{(2\tilde{b}-\tilde{a})^2}{2t}\right) \\
&= \frac{1}{\sqrt{2\pi t}}\exp -\frac{\nu^2 t}{2} + \nu\tilde{a}\left(\exp -\frac{(2\tilde{b}-\tilde{a})^2}{2t}\right).
\end{aligned}$$

Now note that

$$\mathbb{Q}\left\{W_t + \nu t = \tilde{a}, \sup_{0\leq s\leq t}(W_s + \nu s) \geq \tilde{b}\right\} = \mathbb{Q}\{W_t + \nu t = \tilde{a}\}\,\mathbb{Q}\left\{\sup_{0\leq s\leq t}(W_s + \nu s) \geq \tilde{b}\,\middle|\, W_t + \nu t = \tilde{a}\right\}$$

and if Z is a standard Gaussian random variable,

$$\begin{aligned}\mathbb{Q}\{W_t + \nu t = \tilde{a}\} &= \mathbb{Q}\{W_t = \tilde{a} - \nu t\} \\ &= \frac{1}{\sqrt{t}}\mathbb{Q}\left\{Z = \frac{\tilde{a} - \nu t}{\sqrt{t}}\right\} \\ &= \frac{1}{\sqrt{t}}\frac{1}{\sqrt{2\pi}}\exp - \left(\frac{\tilde{a} - \nu t}{\sqrt{t}}\right)^2 /2 \\ &= \frac{1}{\sqrt{2\pi t}}\exp - \frac{(\tilde{a} - \nu t)^2}{2t}.\end{aligned}$$

Hence,

$$\begin{aligned}\mathbb{Q}\left\{\sup_{0\le s\le t}(W_s + \nu s) \ge \tilde{b} \,\middle|\, W_t + \nu t = \tilde{a}\right\} &= \frac{\mathbb{Q}\left\{W_t + \nu t = \tilde{a}, \sup_{0\le s\le t}(W_s + \nu s) \ge \tilde{b}\right\}}{\mathbb{Q}\{W_t + \nu t = \tilde{a}\}} \\ &= \frac{\frac{1}{\sqrt{2\pi t}}\exp - \frac{\nu^2 t}{2} + \nu\tilde{a}\left(\exp - \frac{(2\tilde{b} - \tilde{a})^2}{2t}\right)}{\frac{1}{\sqrt{2\pi t}}\exp - \frac{(\tilde{a} - \nu t)^2}{2t}} \\ &= \exp\frac{2\tilde{b}(\tilde{a} - \tilde{b})}{t}.\end{aligned}$$

Finally, we can compute

$$\begin{aligned}\mathbb{Q}\left\{\sup_{0\le s\le t}(W_s + \nu s) \ge b\right\} &= \int_{\mathbb{R}} \mathbb{Q}\{W_t + \nu t = a\}\,\mathbb{Q}\left\{\sup_{0\le s\le t}(W_s + \nu s) \ge b \,\middle|\, W_t + \nu t = a\right\} da \\ &= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi t}}\exp - \frac{(a - \nu t)^2}{2t}\exp\frac{2b(a - b)}{t} \\ &= e^{-2b\nu}.\end{aligned}$$

□

(e) Deduce the value at time 0 of the option.

Solution

$$\begin{aligned}e^{-rT}\mathbb{Q}\{\sup_{0\le s\le T} W_s + \nu s < b\} &= e^{-rT}(1 - \mathbb{Q}\{\sup_{0\le s\le T} W_s + \nu s \ge b\}) \\ &= e^{-rT}(1 - e^{-2b\nu}).\end{aligned}$$

□

6. Numerical Calculation of Implied Volatility

Compute the implied volatility of a European call option that has an exercise price of \$100, will expire in three months, is worth \$9.50, and the underlying trades at \$103.5. Discount using a short-term risk-free continuously compounding interest rate of 7% per annum.

Solution Below is a C program that uses Newton's method to compute the implied volatility.

Listing 1. Implied Volatility Calculator

```
/*********************************************************************
* Black-Scholes Implied Volatility Calculator
* William Wu, http://themathpath.com
* Date: 2012-01-01
*********************************************************************/
#include <stdio.h>
#include <math.h>
```

```
#include <gsl/gsl_randist.h>
#include <gsl/gsl_cdf.h>
double d1 (double sigma, double s, double e, double t, double r) {
  return (log(s/e) + (r + pow(sigma,2)/2)*t)/(sigma * sqrt(t));
}
double d2 (double sigma, double s, double e, double t, double r){
  return d1(sigma,s,e,t,r) - sigma*sqrt(t);
}
double bs_euro_call (double sigma, double s, double e, double t, double r) {
  return s*gsl_cdf_ugaussian_P(d1(sigma,s,e,t,r)) \
  - e*exp(-r*t)*gsl_cdf_ugaussian_P(d2(sigma,s,e,t,r));
}
double bs_euro_call_volatility_derivative (double sigma, double s, double e,
    double t, double r) {
  return s*gsl_ran_ugaussian_pdf(d1(sigma,s,e,t,r))*(t*(-2*r + pow(sigma,2)) -
      2*log(s/e))/(2*sqrt(t)*pow(sigma,2)) \
  - e*exp(-r*t)*gsl_ran_ugaussian_pdf(d2(sigma,s,e,t,r))*(- (t*(r+pow(sigma,2)
      /2) + log(s/e))/(sqrt(t)*pow(sigma,2)));
}
double implied_volatility_newton_method (double call_price, double sigma_0,
    double s, double e, double t, double r, int num_iters) {
  int i;
  double sigma = sigma_0;
  for (i=0;i<num_iters;i++)
    sigma = sigma - ((bs_euro_call(sigma,s,e,t,r)-call_price)/
        bs_euro_call_volatility_derivative(sigma,s,e,t,r));
  return sigma;
}
int main(int argc, char** argv) {
  double call_price = 9.5;
  double sigma_0 = 0.5;
  double s = 103.5, e = 99, t = 0.25, r = 0.07;
  int num_iters = 100;
  printf("%lf\n",implied_volatility_newton_method (call_price,sigma_0,s,e,t,r,
      num_iters));
  return 0;
}
```

Plugging in the following parameters

programming variable	description	value
s	price	103.5
e	strike	99
t	maturity time	0.25
r	rate	0.07
call_price	call price	9.5

yields an implied volatility of $\boxed{0.293756}$.

□

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